Lecture Notes on The Langlands Conjecture

IAS Workshop

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Contents

Introduction

Lecturers

1 Overview of algebraic geometry

Here I have collected a few facts from algebraic geometry.

1.1 Algebraic varieties and rings

The basic idea in algebraic geomtry is to describe the geometry of a space Ξ in terms of the properties of the set of functions on that Ξ. The set of funnctions forms a **ring**, which more or less means that we can add and multiply functions (pointwise) and also multiply functions by scalars. Thus, an algebraic variety corresponds to a ring. Now let

$$X{:=}\{f : \Xi \to \mathbb{R}.\}$$

be such a ring. How do we recover the points of Ξ from the algebraic structure X?

The idea is that each point $x \in \Xi$ defines a subset

$$I_x{:=}\{f \in X| \quad f(x) = 0\} \subset X.$$

This subset is an **ideal** which means that it is a *ring* by itself and furthermore if we multiply an element $f \in I_x$ by an element $g \in X$ we get an element $f \cdot g \in I_x$ in the *ideal.*

Now let's turn things around. Suppose we are given an ideal $I \subset X$. Is it of the form I_x? In other words, does it correspond to a point $x \in X$? The general answer is no. To actually correspond to a point, the ideal has to be a **maximal ideal**, which means that if we try to make the ideal any bigger it will have to be all of X. ($I \subset I' \subseteq X \implies I' = X$.)

So, in algebraic geometry we start with a *ring* X and define the associated variety as the set of maximal ideals of X, called the **spectrum** $\mathrm{Spec}(X)$ of X. Alternatively, we also denote it by $X(\mathbb{R})$ if we work with real-valued functions, or $X(\mathbb{C})$ if we work with complex valued functions.

1.2 Vector bundles and modules

A vector bundle is, in the language of algebraic geometry, "a locally free module over the structure sheaf."

1.3 Local system

This is taken from [Deligne]: "Given an algebraic variety X over a number field $\mathbb{F}$, let $\pi_1 = \pi_1(X(\mathbb{C}), p)$ be the topological fundamental group of the set of complex points $X(\mathbb{C})$ with some chosen (algebraic) base point p. Then, there is a bijection between finite dimensional linear representations $\rho : \pi_1 \to GL_n(\mathbb{C})$ and rank-n **local systems** on $X(\mathbb{C})$, i.e. complex analytic vector bundles of rank n over $\mathbb{C}$ equipped with an integrable (i.e. flat) connection."

Lectures given by **Mark Goresky**.

2 Monday morning session

As a precaution, all our statements will be approximately correct. We can make exactly correct but unintelligeble statements, or false and clear ones. We will choose the latter. The plan is as follows.

1. Approximate statement of the conjecture.
2. Review of **fields** and **Galois groups**.
3. Galois representations.
4. Automorphic forms for $SL(2)$.
5. Hecke operators.
6. Hecke algebras.
7. General comments.

2.1 Statement of the Laglands conjecture

We will first state the Laglands conjecture and then explain all the ingredients. The **Langlands conjecture** is a proposed equivalence

"Nice" irreducible n-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$	$\Longleftrightarrow$	"nice" cuspidal automorphic representations of $GL_n(\mathbb{A}_{\mathbb{Q}})$

Such that

Eigenvalues of Frob_p	$\Longleftrightarrow$	eigenvalues of Hecke operators at p

Here n is a positive integer, Gal is the **Galois group** that will be explained later, $\mathbb{A}_{\mathbb{Q}}$ are the **Adeles** over the rational numbers $\mathbb{Q}$, p is a prime integer, and Frob_p is the Frobenius element.

2.2 Fields

Let $E \subset F$ be a field extension. Then, the **Galois group** is defined as

$$\mathrm{Gal}(F/E) = \{\text{field automorphisms of } F \text{ that fix every } e \in E\}.$$

Example. For $\mathbb{R} \subset \mathbb{C}$ we have $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \tau\}$, where τ is complex conjugation. The Galois group in this case is isomorphic to $\mathbb{Z}(2)$.

Let

$$f_1(x_1, x_2, \ldots, x_n), \cdots, f_r(x_1, x_2, \ldots, x_n),$$

be polynomials with coefficients in the field E. Then

$$X(E){:=}\{(\mathbf{x})|f_i(\mathbf{x}) = 0, \quad i = 1 \ldots r\}, \qquad \mathbf{x}{:=}(x_1, x_2, \ldots, x_r),$$

defines an **algebraic variety**. [Here we use the notation whereby $(\mathbf{y})$ denotes the ideal in the ring of polynomials $E[x_1, \ldots, x_n]$ of all polynomials that vanish at $\mathbf{y}$, for some $\mathbf{y} \in E^n$. In algebraic geometry, such ideals are identified with the points of E^n, so we use $(\mathbf{y})$ and $\mathbf{y}$ interchangeably.] The polynomials $f_1, \ldots, f_r$, also define an algebraic variety in the extension field F,

$$X(F){:=}\{\mathbf{y}|f_i(\vec{x} = 0, \quad i = 1 \ldots r\}, \qquad \mathbf{x}{:=}(x_1, x_2, \ldots, x_r),$$

We call this the **F-points** of X. $X(E)$ is an algebraic variety defined over E. $\mathrm{Gal}(F/E)$ acts on $X(F)$ in a way that fixes the subvariety $X(E)$. We will now discuss the various fields that appear in what follows.

2.3 Finite fields

indexfinite fieldsFor every prime number p and for every $n \geq 1$ there exists a unique field $\mathbb{F}(p^n)$ with p^n elements. Furthermore, we have

$$\mathbb{F}(p^n) \subset \mathbb{F}(p^m) \qquad \Longleftrightarrow n|m \quad (n \text{ divides } m).$$

Set

$$q{:=}p^n.$$

Then $\mathrm{Gal}(\mathbb{F}(q^n)/\mathbb{F}(q)) \simeq \mathbb{Z}(r)$ is a cyclic group, [where $r = (q^n - 1)/(q - 1)$], and is generated by the **Frobenius element**,

$$\mathrm{Frob}_q \quad (x \mapsto x^q), \qquad \mathrm{Frob}_q \in \mathrm{Gal}(\mathbb{F}(q^n)/\mathbb{F}(q)).$$

For example, consider $\mathbb{F}(p^r)/\mathbb{F}(p)$, where p is prime. Then $\mathbb{F}(p)$ is identified with $\mathbb{Z}(p)$, the integers modulo p. [In any ring of characteristic p the following identity holds,]

$$(x+y)^p \equiv x^p + \binom{p}{1} x^{p-1} y + \cdots + y^p \equiv x^p + y^p \pmod{p}.$$

So $x \mapsto x^p$ is an additive map on all of $\mathbb{F}(p^n)$. It also fixes $\mathbb{F}(p) = \mathbb{Z}(p)$ since $m^p \equiv m \pmod{p}$ for all integers $m \in \mathbb{Z}$.

Now consider $\mathbb{F}(q)$ for $q = p^n$. All elements $x \in \mathbb{F}(q)$ satisfy $x^q = x$. In fact $x^q - x$ completely splits over $\mathbb{F}(q)$,

$$x^q - x = \prod_{a \in \mathbb{F}(q)} (x - a).$$

2.4 Number fields

A **number field** is a finite field extension of the rational numbers $\mathbb{Q}$. For any polynomial with rational coefficients $f(x) \in \mathbb{Q}[x]$, the solutions of $f(x) = 0$ lie in some number field. For example, for $x^2 + 1$ the number field is

$$\mathbb{Q}[i] \equiv \{a + b\sqrt{-1} | \quad a, b \in \mathbb{Q}\}.$$

The integers $\mathbb{Z} \subset \mathbb{Q}$ have the property that the entire number field $\mathbb{Q}$ can be written as

$$\mathbb{Q} = \{\tfrac{a}{b} | \quad a, b \in \mathbb{Z}\}.$$

This notion of an "integer" can be extended to any number field. For any number field E the **ring of integers** is defined to be the [minimal] subring $\mathcal{O}_E \subset E$ such that

$$E = \{\tfrac{a}{b} | \quad a, b \in \mathcal{O}_E\}.$$

Every number field can be embedded in $\mathbb{C}$. The union of all the number fields, as subfields of $\mathbb{C}$, is called the **closure of** $\mathbb{Q}$ and denoted by

$$\overline{\mathbb{Q}} := \bigcup_{\text{number fields}} E \subset \mathbb{C}.$$

2.5 p-adic fields

The p**-adic!integers** are defined as formal infinite sums,

$$\mathbb{Z}_p := \{a_0 + a_1 p + a_2 p^2 + \cdots | \qquad 0 \leq a_i \leq p\}.$$

We add and multiply p-adic integers *with carry.*

There is a ring homomorphism

$$\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/(p),$$
$$\sum a_j p^j \mapsto a_0,$$

(Recall that '$\twoheadrightarrow$' means a **surjective map**.) The p-adic integers $\mathbb{Z}_p$ actually contains many non-integer elements of $\mathbb{Q}$! For example, $\frac{1}{3} \in \mathbb{Z}_5$ as follows

$$\frac{1}{3} = 2 + 3 \cdot 5 + 1 \cdot 5^2 + 3 \cdot 5^3 + 1 \cdot 5^4 + \cdots$$

which can be checked by calculating (recall that we multiply with carry),

$$3 \times (2 + 3 \cdot 5 + \cdots) = (1 + 5) + (4 + 5) \cdot 5 + \cdots = 1 + 0 \cdot 5 + 0 \cdot 5^2 + \cdots$$

[Note that the formal geometric series $1 + 5^2 + 5^4 + \cdots$ equals $-1/24$.] In fact, $\mathbb{Z}_p$ contains all fractions $\frac{a}{b}$ such that b is not divisible by p. More generally,

$$\sum_{j=0}^{\infty} a_j p^j \text{ is invertible} \Longleftrightarrow a_0 \neq 0.$$

So we define the **p-adic rationals** as

$$\mathbb{Q}_p := \{ \tfrac{a}{b} | \quad a, b \in \mathbb{Z}_p \}.$$

Alternatively, we can define it as the set of formal Laurent series in p,

$$\mathbb{Q}_p := \{ \sum_{j=m}^{\infty} a_j p^j | \qquad 0 \leq a_j \leq p, \quad m > -\infty \}.$$

The p-adic rationals $\mathbb{Q}_p$ form a *field.*

Remark 2.1. Even though there is a map $\mathbb{Z}_p \to \mathbb{Z}/(p)$, the field $\mathbb{Q}_p$ has characteristic 0. The characteristic of a field is defined as the smallest number c such that $cx = 0$ for *all* x.

$\mathbb{Q}_p$ is a **completion** of $\mathbb{Q}$. In fact there is a **p-adic topology on** $\mathbb{Q}$, that is induced from a norm. [The norm is defined as follows. To calculate $\|\frac{a}{b}\|_p$, write $\frac{a}{b} = p^k \frac{c}{d}$ for some integer $k \in \mathbb{Z}$ and such that c, d are not divisible by p. The set $\|p^k \frac{c}{d}\|_p = p^{-k}$.] In this topology, the "completion" refered to above is just the usual completion in the topological sense, i.e. the set of all limits of Cauchy sequences.

So, we see that $\mathbb{Q}$ has different completions: $\mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_5, \mathbb{Q}_7, \mathbb{Q}_{11}, \ldots$ The fields $\mathbb{Q}_p$ are also called the **p-adic fields.**

2.6 Function fields

Let X be a Riemann surface. Denote by $\mathbb{C}(X)$ the **meromorphic functions** (also called **rational functions**) on X. From the algebraic geometry point of view [a meromorphic function on X defines an embedding] $X \subset \mathbb{P}^2(\mathbb{C})$. We can extend these definitions for finite fields.

Take a nonsingular one-dimensional algebraic variety $X \subset \mathbb{P}^2(\mathbb{F}_q)$.

Remark 2.2. The algebraic variety is identified as the set of polynomial equations that define X (more precisely, the ideal). It is not just the set of points that solve the polynomial equations. The distinction is not so important for $\mathbb{C}$ which is algebraically closed, but it is important for $\mathbb{F}_q$, which is not algebraically closed.

We denote by $\mathbb{F}_q(X)$ the **rational functions on** X [defined as the field of fractions]. This is an example of a **global function field.**

In the context of $\mathbb{Q}$ and $\mathbb{Q}_p$, the field $\mathbb{Q}$ is *global* and $\mathbb{Q}_p$ is a **local field**.

The function field $\mathbb{F}_q(X)$ also has its own extensions and corresponding Galois groups. By the way, the Galois group $\mathrm{Gal}(\mathbb{Q}_p/\mathbb{Q})$ is complicated, and will not be discussed here.

Over $\mathbb{C}$, the following identification will be very important later,

$$\mathrm{Gal}(\overline{\mathbb{C}(X)}/\mathbb{C}(X)) = \pi_1(X)$$

where $\pi_1(X)$ is the well-known fundamental group of X from algebraic topology. This connection can be gotten from the universal cover $Y \to X$. [To see the connection between $\pi_1(X)$ and the Galois group above, note that $\overline{\mathbb{C}(X)}$ can be identified with $\mathbb{C}(Y)$ where Y is the universal cover of X. The fundamental group $\pi_1(X)$ can be identified with the group of **deck transformations** of the cover Y. Given a representative loop γ for an element in $\pi_1(X)$ and a point $y \in Y$ one can lift γ into a continuous path that starts at y. In general, the path might not necessarily end at the same point y, but it will end at a point in the same fiber of the universal cover Y over X. $\pi_1(X)$ therefore acts on Y and hence on the field of meromorphic functions $\mathbb{C}(Y)$. This action fixes $\mathbb{C}(X) \subset \mathbb{C}(Y)$, where $\mathbb{C}(X)$ is identified with the functions on the cover Y that depend only on the point on the base X. So this action is an element of $\mathrm{Gal}(\mathbb{C}(Y)/\mathbb{C}(X))$.]

Function fields also have **completions**. The completion of $\mathbb{C}(X)$ at $x \in X$ is the field of formal Laurent series

$$\mathbb{C}((t)) := \{ \sum_{i > -\infty} a_i t^i | \quad a_i \in \mathbb{C} \}$$

here t is a local coordinate near the point $x \in X$.

Similarly, for a nonsingular algebraic curve X, the completion of the rational function field $\mathbb{F}_q(X)$ at a point $x \in X(\mathbb{F}_q)$ is denoted by $\mathbb{F}_q((t))$. It can be identified with formal series

$$F_q((t)) \simeq \{ \sum_{i>-\infty} a_i t^i | \quad a_i \in \mathbb{F}_q(X)\}.$$

These are **local function fields**.

For a local function field, we can define the **ring of integers** as the ring of formal power series,

$$\mathbb{C}[[t]]:=\{\sum_{i=0} a_i t^i | \quad a_i \in \mathbb{C}\}.$$

Similarly,

$$\mathbb{F}_q[[t]]:=\{\sum_{i=0} a_i t^i | \quad a_i \in \mathbb{F}_q\}.$$

Note that every formal Laurent series in $\mathbb{C}((t))$ can be expressed as a fraction $\frac{a}{b}$ with $a, b \in \mathbb{C}[[t]]$.

These local function fields at different points $x \in X(\mathbb{F}_q)$ (or $x \in X(\mathbb{C})$) are all isomorphic. The only dependence on x is through the what the global field is embedded inside the local field. We have the injective maps

$$\mathbb{C}(X) \hookrightarrow \text{completion}, \qquad \mathbb{Q} \hookrightarrow \mathbb{Q}_p, \qquad \forall p$$

$$F_q(X) \hookrightarrow \text{completion}, \qquad \mathbb{Q} \hookrightarrow \mathbb{R}.$$

2.7 Etale cohomology

Etale cohomology is a construction that naturally produces lots of Galois representations. Let X be an algebraic variety over a field E. From this data one can define $H^i_{\text{et}}(X; \mathbb{Q}_l)$ – the Etale cohomolgy of X. Its coefficients are p-adic numbers [Deligne-Grothendik]. All the usual theorems of cohomology such as the Myers-Vietoris sequence, the Künneth formula, etc. can be extended to H^*_{et}. H^i_{et} is a vector space that depends on l, but we will pretend that it doesn't. (For most l it is true.)

2.8 Galois representations

$\text{Gal}(\overline{E}/E)$ acts on $H^i_{\text{et}}(X; \mathbb{Q}_l)$. This is the most common way that one encounters Galois representations. For example, suppose that Y is a nonsingular projective algebraic variety defined over $\mathbb{F}_q$, with $q = p^n$. Then $H^i_{\text{et}}(Y; \mathbb{Q}_l)$ is defined for $l \neq p$. The Frobenius element $\text{Frob}_q \in \text{Gal}(\overline{E}/E)$ acts on the representation.

$$H^*_{\text{et}} \curvearrowleft \text{Frob}_q\,.$$

It is a fact that

$$H^i_{\text{et}} = 0, \qquad \text{for } i > 2\dim X.$$

The eigenvalues α of Frob_q on $H^{2i}_{\text{et}}(Y)$ have [l-adic] norm $\|\alpha\| = q^i$. The eigenvalues β of Frob_q on $H^{2i+1}_{\text{et}}(Y)$ have [l-adic] norm $\|\beta\| = q^i\sqrt{q}$.

The Etale cohomology has enough information to recover a finite covering of Y by open sets [???].

There is a formula for Etale cohomology that is reminiscent of the **Lefshetz fixed point formula** from algebraic geometry and relates the number of points in the variety to the eigenvalues α_k and β_k:

$$\#Y(\mathbb{F}(q^r)) = \sum_k \alpha_k^r - \sum_k \beta_k^r. \tag{1}$$

Thus, if we know the numbers $\#Y(\mathbb{F}(q^r))$ for sufficiently many r's, we have enough equations for solve for the eigenvalues α's and β's. In particular since $\|\alpha_k\|$ uniquely determines the i of H^i_{et} we get recover the dimensions of H^i_{et} In particular,

$$\operatorname{rank} H^{2i}_{\text{et}}(Y) = (\#\text{ of }\alpha\text{'s such that }\|\alpha\| = q^i).$$

These eigenvalues came up in the Langlands conjecture. One can encode them in an L-function, which is a generalization of a Riemann ζ-function. Define

$$\zeta_q^{(Y)}(t) := \frac{\prod_{\text{odd } i} \det(\mathbb{I} - t\,\text{Frob}_q) \quad \text{on } H^i_{\text{et}}(Y; \mathbb{Q}_l)}{\prod_{\text{even } i} \det(\mathbb{I} - t\,\text{Frob}_q) \quad \text{on } H^i_{\text{et}}(Y; \mathbb{Q}_l)}$$

The last identity follows immediately from (1). The L-function satisfies a functional equation which is a generalization of **Poincaré duality**

$$\zeta_q^{(Y)}(\tfrac{1}{q^d t}) = \pm q^{\frac{d}{2}\chi(Y)} t^{\chi(Y)} \zeta_q^{(Y)}(t)$$

Here $\chi(Y)$ is the **Euler characteristic** of Y and $d = \dim(Y)$.

Example. The complex projective space can be defined as

$$\mathbb{CP}^{n-1} = (\mathbb{C}^n - \{0\})/\mathbb{C}^*, \qquad \mathbb{C}^* = \mathbb{C} - \{0\}.$$

Similarly, over a finite field we define

$$\mathbb{P}^{n-1}(\mathbb{F}_q) := (\mathbb{F}_q^n - \{0\})/\mathbb{F}_q^*.$$

Counting the number of points in this variety we have

$$\#(\mathbb{P}^{n-1}(\mathbb{F}_q)) = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}.$$

This is exactly the Poincaré polynomial of projective space.

Thus far we have considered a variety Y that is defined over a finite field. Now suppose that Y is defined over $\mathbb{Z}$. Then $Y \pmod p$ makes sense.

Theorem 2.1.

$$H^i_{et}(Y_{\bmod p}; \mathbb{Q}_l) \cong H^i(Y(\mathbb{C}); \mathbb{Q}) \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_l,$$

where H^i is ordinary (Čech) cohomology, and $\cong$ denotes an isomorphism of vector spaces.

Thus, if Y comes from a variety over $\mathbb{Z}$, all the information in $H^*_{\mathrm{et}}(Y)$ is already present in the ordinary Čech cohomology $H^i(Y, \mathbb{Q})$. But note that we still get a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^i_{\mathrm{et}}(Y; \mathbb{Z}_l)$, which is a special feature of Etale cohomology.

We can now put it all together and define the **Hasse-Weil L-function** of Y

$$L(Y, s) := \prod_{p \text{ prime}} \zeta_p(Y, p^{-s}).$$

We note that there is a problem when $p = l$, but we will not get into that.

One can also define L-functions for a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We have the embeddings

$$\mathbb{Q} \hookrightarrow \mathbb{Q}_p, \qquad \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p,$$

where $\overline{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$. So, we get natural maps

$$\begin{tikzcd}
\mathrm{Frob}'_p \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \arrow[r, hook] \arrow[d, two heads] & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \ni \mathrm{Fr}_p \\
\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) &
\end{tikzcd}$$

The map $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is surjective, so starting from the Frobenius element $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ we can find a lift $\mathrm{Frob}'_p \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. The map $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is $1 - 1$ so we can map the lift Frob_p' to an element $\mathrm{Fr}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus, we get the **Frobenius element at** p

$$\mathrm{Fr}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

[Is it obviously unique up to conjugation?]

Pick a representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. (If you are lucky it will descend to $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ [?]. We can consider

$$\zeta_p(t) = \frac{1}{\det(\mathbb{I} - t\,\mathrm{Frob}_p) \quad \text{on "}V\text{"}}.$$

Skipping technicalities involving what is called the **inertia group**, we define

$$L_V(s) = \prod_{p \text{ prime}} \zeta_p(p^{-s}).$$

For example, if we start with the trivial representation, we get

$$L(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \zeta(s),$$

the **Riemann ζ-function**. We have just seen that it is also the L-function for the trivial representation of the Galois group of $\overline{\mathbb{Q}}$.

2.9 Recap

Let's recap what we have so far. We are trying to study irreducible n-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and relate them to cuspidal automorphic representations of $GL_n(\mathbb{A}_{\mathbb{Q}})$ (that have not been defined yet). We had the fields

	Global fields	$\mathbb{Q}$,	$\mathbb{F}_q(X)$	
Completions ⇒	Local fields	$\mathbb{Q}_p$,	$\mathbb{F}_q((t))$	(Laurent series)
	Ring of integers	$\mathbb{Z}_p$,	$\mathbb{F}_q[[t]]$	

The rings of integers had **quotients** $\mathbb{F}_p$ and $\mathbb{F}_q$ [???]. We defined the Frobenius elements Frob_p and Frob_q respectively. Frob_p lifts to an element $\mathrm{Fr}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and similarly Frob_q lifts to an element $\mathrm{Fr}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q(X)/\mathbb{F}_q(X))$.

Given a representation of the Galois group we can ask for the eigenvalues of Fr_p. Note that $\overline{\mathbb{Q}}$ is the field of **algebraic numbers** algebraic numbers and is a subfield of $\mathbb{C}$. Also note that the eigenvalues of Fr_p are complex numbers.

2.10 The Adeles of $\mathbb{Q}$

The **Adeles of** $\mathbb{Q}$ are defined as the *ring*

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p = \{a_\infty, a_2, a_3, a_5, a_7, a_{11}, \dots,$$
$$a_\infty \in \mathbb{R}, \quad a_p \in \mathbb{Q}_p, \quad \text{but } \exists N \text{ such that } a_p \in \mathbb{Z}_p \quad \forall p \geq N\}.$$

The Adeles are an object of central importance in number theory. We define

$$GL_n(\mathbb{A}_{\mathbb{Q}}) := \text{invertible } n \times n \text{ matrices with entries in } \mathbb{A}_{\mathbb{Q}}.$$

The **automorphic representations** that we'll be looking at are representations of $GL_n(\mathbb{A}_{\mathbb{Q}})$ with certain properties.

Functions on $GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_{\mathbb{Q}})$ arise from **modular forms**. Note that to define the coset $GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_{\mathbb{Q}})$, we need to specify the embedding $\mathbb{Q} \subset \mathbb{A}_{\mathbb{Q}}$. We choose the diagonal embedding. (Using the embeddings $\mathbb{Q} \hookrightarrow \mathbb{R}$ and $Q \hookrightarrow \mathbb{Q}_p$.) We can write

$$GL_n(\mathbb{A}_{\mathbb{Q}}) = GL_n(\mathbb{R}) \times \prod_{p \text{ prime}}{}' GL_n(\mathbb{Q}_p).$$

2.11 Automorphic representations

A "good" representation π of $GL_n(\mathbb{A}_{\mathbb{Q}})$ decomposes as a product of representations of the individual factors

$$\pi = \pi_\infty \bigotimes_p \pi_p, \tag{2}$$

a product of representations of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{Q}_p)$. We are interested in those representations for which each representation π_p of $GL_n(\mathbb{Q}_p)$ has a 1-dimensional subspace that is fixed under $GL_n(\mathbb{Z}_p)$.

A vector in this fixed subspace defines a function of the **double coset**,

$$GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_{\mathbb{Q}}) / \prod GL_n(\mathbb{Z}_p) \times SO(n).$$

Note that $GL_n(\mathbb{Z}_p)$ is the maximal compact subgroup of $GL_n(\mathbb{Z}_p)$ in the p-adic topology, like $SO(n)$ is the maximal compact subgroup of $GL_n(\mathbb{R})$.

$GL_n(\mathbb{A}_{\mathbb{Q}})$ acts on $GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_{\mathbb{Q}})$ and therefore acts on the vector space of (complex valued) functions $\mathbb{C}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_{\mathbb{Q}}))$. It is not quite true that $\mathbb{C}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_{\mathbb{Q}}))$ decomposes into irreducible representations, but it is almost true. Let

$$\pi := \text{An irreducible subspace of } \mathbb{C}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_{\mathbb{Q}})) \text{ preserved under } GL_n(\mathbb{A})$$

We restrict to such a representation π. It has a decomposition (2). Now consider only those representations π which for each p have a unique 1-dimensional subspace of vectors fixed under $GL_n(\mathbb{Z}_p)$ in each factor and fixed under $SO(n)$ for the first factor. (Actually, it is sufficient to demand this for almost all p, with special consideration to the special p's for which this does not hold. To simplify things, we will assume that for *all* p we have such a fixed vector (**unramified representations**).

2.12 The double coset

We have the double coset space

$$GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}_\mathbb{Q})/SO(n) \times \prod_{p \text{ prime}}{}^{'} GL_n(\mathbb{Z}_p).$$

In the Langlands correspondence,

- on the "Galois side" we have Frobenius eigenvalues of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- on the "automorphic side" we will look at Hecke eigenvalues of automorphic representations of $GL_n(\mathbb{A})$.

These two sets of eigenvalues will correspond to each other. Let's now discuss the double coset. Things are somewhat easier for SL_2 so we will first take $n = 2$. Following the definitions, one can show that

$$SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}_\mathbb{Q})/SO(2) \times \prod_{p \text{ prime}}{}^{'} SL_2(\mathbb{Z}_p) = SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO(2).$$

[To see this, use the fact that in the p-adic topology $SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{Q}_p)$, so one can write a matrix $M \in SL_2(\mathbb{Q}_p)$ as $M = M_{\mathrm{sl}} + M_{\mathrm{Z}}$ with $M_{\mathrm{sl}} \in SL_2(\mathbb{Q})$ and $M_{\mathrm{Z}} \in p^k\mathbb{Z}_p$ for some arbitrarily big $k \in \mathbb{N}$. Also, the elements of M_{sl} are all of the form a/p^s for some $s \in \mathbb{N}$, and $a \in \mathbb{Z}$ not divisible by p. Thus $M_{\mathrm{sl}} \in \mathbb{Z}_{p'}$ for all $p' \neq p$.]
We can identify

$$SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO(2) \simeq SL_2(\mathbb{Z})\backslash \mathbb{U}$$

where $\mathbb{U}$ is the **upper half plane**. An analytic function on $SL_2(\mathbb{Z})\backslash\mathbb{U}$ is a **modular form**.

For a second example, define

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \middle| \quad c \equiv 0 \pmod{N} \right\},$$

and consider

$$Z(\mathbb{A}_\mathbb{Q}) \times SL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q})/O(2) \times K$$

where $Z(\mathbb{A}_{\mathbb{Q}})$ is the center comprising of diagonal scalar matrices

$$Z(\mathbb{A}_{\mathbb{Q}}) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \middle| \alpha \in \mathbb{A}_{\mathbb{Q}} \right\}$$

and

$$K := \prod_{p \nmid N} GL_2(\mathbb{Z}_p) \times \Gamma_0(N).$$

As before, one can show that

$$Z(\mathbb{A}_{\mathbb{Q}}) \times SL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_{\mathbb{Q}})/O(2) \times K = \Gamma_0(N) \backslash \underbrace{SL_2(\mathbb{R})/O(2)}_{\mathbb{U}}.$$

There is a technical condition that requires **cuspidal forms** that we will not get into.

2.13 Hecke algebra

Define the **Hecke algebra** $\mathcal{H}_p$ to be the ring of finite formal linear combinations of double cosets

$$\mathcal{H}_p := \{\Gamma g \Gamma \in GL_n(\mathbb{Z}_p) \backslash GL_n(\mathbb{Q}_p)/GL_n(\mathbb{Z}_p)\}, \qquad \Gamma \equiv GL_n(\mathbb{Z}_p).$$

This can be turned into an algebra by defining the product as follows. It turns out that the product of two cosets can be written as a disjoint union of cosets

$$(\Gamma g \Gamma)(\Gamma g' \Gamma) = \bigsqcup_h \Gamma h \Gamma$$

with finitely many h's on the right hand side. We then define the product in the Hecke algebra as

$$(\Gamma g \Gamma) \cdot (\Gamma g' \Gamma) := \sum_h \Gamma h \Gamma$$

It is also a fact that the Hecke algebra $\mathcal{H}_p$ is commutative. The Hecke algebra $\mathcal{H}$ breaks up into a product

$$\mathcal{H} = \bigotimes_p \mathcal{H}_p,$$

where $\mathcal{H}_p$ was defined above.

Now suppose we have one of the representations from section (2.11) with the 1-dimensional subspace preserved under $GL_n(\mathbb{Z}_p)$. Then $\mathcal{H}_p$ acts on it. The action is given by an *integral* over the coset. That is,

$$(\Gamma g \Gamma) x \mapsto \int_\Gamma \Gamma g x, \qquad x \in \pi_p, \qquad g \in GL_n(\mathbb{Q}_p), \qquad \Gamma \equiv GL_n(\mathbb{Z}_p).$$

It works because there is a *unique* Γ-invariant 1-dimensional vector subspace in π_p and therefore $\int \Gamma g x$, which is Γ-invariant must be in that subspace.

Now let $\pi = \pi_\infty \otimes_p \pi_p$ and assume that there is a 1-dimensional subspace in π_p that is fixed by $GL_n(\mathbb{Z}_p)$. Then $\mathcal{H}_p$ acts on π.

Theorem 2.2. *The* 1*-dimensional representations of* $\mathcal{H}_p$ *are in one-to-one correspondence with diagonal matrices*

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, \qquad a_i \in \mathbb{C} \textit{ defined up to permutation.}$$

(The a_i's can be thought of as the "moduli" of the 1-dimensional representation.)

Theorem 2.3.

$$\mathcal{H}_p \cong \mathbb{C}[\mathcal{T}/W] \equiv \textit{space of symmetric polynomials on } \mathbb{C}^n.$$

where $\mathcal{T}$ *is a* **maximal torus** *of* GL_n *and* W *is the* **Weyl group** *of* GL_n.

It follows that from an irreducible representation of $GL_n(\mathbb{A})$ (satisfying some requirements of we we mentioned a few above) one can construct for each p the invariants

$$\pi \rightsquigarrow (a_1, \dots, a_n), \qquad \text{defined up to permutations.}$$

The $a_1, \dots, a_n$ are called **Hecke eigenvalues**.

Conjecture 2.4. *For this representation* π *there is a corresponding Galois representation and the eigenvalues* a_i *of the representation should correspond to the Hecke eigenvalues of* π.

The conjecture can also be stated as an equality of the L-functions. The Hecke eigenvalues $a_1, \dots, a_n$ can be encoded in an L**-function**, and, according to the conjecture, there should exist a Galois representation whose L-function is the same. We can summarize this in the diagram,

Frobenius eigenvalues at p $\Longleftrightarrow$ Hecke eigenvalues

n-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ $\Longleftrightarrow$ Automorphic representations of $GL_n(\mathbb{A})$

For SL_2, it can be shown that the **classical modular forms** T_p generate the Hecke algebra which is the algebra of the double cosets of $SL_2(\mathbb{Z})\backslash GL_2(\mathbb{Q})/SL_2(\mathbb{Z})$ with $\det > 0$. [???] The space $SL_2(\mathbb{Z}_p)\backslash GL_2(\mathbb{Q}_p)/SL_2(\mathbb{Z}_p)$ is generated by $\{T_{(p^n)}\}$. [???]

According to the Atkins-Shimura theorem [Atkins-Shimura] [?] automorphic representations give rise to auromorphic forms. An **automorphic (modular) form** is a function on

$$SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO(2).$$

It lifts to a function on $SL_2(\mathbb{A})$. We can then "move it around" by the right action [of $SO(2) \times \prod'_p SL_2(\mathbb{Z}_p)$] and take linear combinations of the resulting functions to get a representation.

The Etale cohomology with coefficients in a certain group $\mathcal{E}_k$ breaks up into a direct sum,

$$H^1_{\text{et}}(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}_\mathbb{Q})/SO(2) \times \prod_{p \text{ prime}}{}' SL_2(\mathbb{Z}_p); \mathcal{E}_k) \cong \{\text{cuspidal modular forms}\} \oplus \overline{\{\text{cuspidal modular forms}\}},$$

where $\overline{\{\cdots\}}$ denotes the complex conjugate representation. We cannot write a similar formula for $n = 3$ because, while

$$SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}_\mathbb{Q})/SO(2) \times \prod_{p \text{ prime}}{}' SL_2(\mathbb{Z}_p)$$

turns out to be an algebraic variety,

$$SL_3(\mathbb{Q})\backslash SL_3(\mathbb{A}_\mathbb{Q})/SO(3) \times \prod_{p \text{ prime}}{}' SL_3(\mathbb{Z}_p)$$

is not!

Lectures given by **Kari Vilonen**.

3 Monday afternoon session

3.1 Langlands conjecture for function fields

Now we will stay exclusively in the function field case. Recall our notation

$$q = p^n, \qquad l \neq p.$$

Let X be a smooth proper compact curve over $\mathbb{F}_q$. We denote

$$F \equiv \mathbb{F}_q(X).$$

Let $Y \to X$ be an unramified covering. Denote

$$F_{\text{un}} \equiv \mathbb{F}_q(Y).$$

Then

$$\text{Aut}(F_{\text{un}}/F) = \pi_1(X),$$

where Aut is the group of automorphisms [Same as Gal here?] and $\pi_1(X)$ is the **fundamental group**. In fact, in algebraic geometry, we can take this equation as the *definition* of $\pi_1(X)$.

So we have the Laglands correspondence,

$$\sigma \in \{\text{Irreducible representations } \pi_1(X) \to GL_n(\overline{\mathbb{Q}}_l)\}$$
$$\rightsquigarrow \{\text{cuspidal functions on } GL_n(F)GL_n(\mathbb{A})/GL_n(\mathcal{O})\} \ni f_\sigma.$$

Our notations is as follows. At every point $x \in |X|$ [Notation?] we can consider the completion

$$\mathbb{F}_{q\,x}((t_x)) \equiv V_x \supset \mathcal{O}_x \equiv \mathbb{F}_{q\,x}[[t_x]],$$

(where t_x is a local coordinate at x), and we define

$$\mathbb{A} := \prod_{x\in|X|}{}' F_x \supset \mathcal{O} = \prod_{x\in|X|} \mathcal{O}_x.$$

The $\prod'$ denotes a **restricted product**, i.e. for almost all $x \in |X|$ the corresponding element in F_x is required to be in $\mathcal{O}_x \subset F_x$.

Then, for each irreducible representation σ of $\pi_1(X)$ we associate a cuspidal function f_σ on $GL_n(F)GL_n(\mathbb{A})/GL_n(\mathcal{O})$.

3.2 Hecke operator

Theorem 3.1. *(Laforque's theorem [?]) For each $x \in |X|$, and $i = 1 \dots n$, we have an integral operator*

$$T_x^i : GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O}) \to GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O}).$$

such that

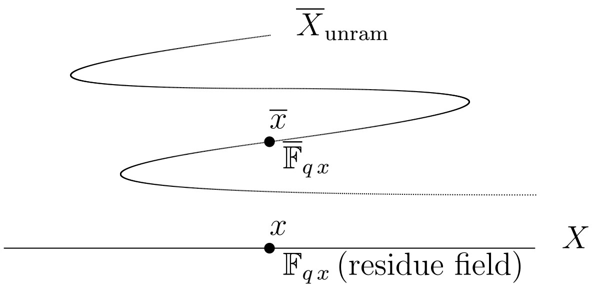
$$T_x^i f_\sigma = b_{x,i} f_\sigma$$

where

$$b_{x,i} = i^{th} \textit{ elementary symmetric function of the eigenvalues of } \sigma(\mathrm{Fr}_x).$$

Recall that for each $x \in |X|$ we got in section (2.8) a Frobenius element $\mathrm{Fr}_x \in \mathrm{Gal}(\overline{\mathbb{F}}_x/\mathbb{F}_x)$, up to conjugacy. In our case $\mathrm{Fr}_x \in \pi_1(X)$.

Remark 3.1. The situation here does not have an analog for varieties over $\mathbb{C}$ because $\mathbb{C}$ is algebraically closed so that $\overline{\mathbb{C}} = \mathbb{C}$. If we have an unramified cover $\overline{X}_{\text{unram}}$ then the **residue field** at $x \in |X|$ is the same as the residue field at a lift $\overline{x} \in |\overline{X}_{\text{unram}}|$. But for finite fields, the situation is different. This is depicted in the drawing:



Despite all that, in the rest of the discussion we will restrict to varieties over $\mathbb{C}$ afterall!

3.3 Appearance of vector bundles

So from now on we replace $\mathbb{F}_q$ by the field $\mathbb{C}$. (Although the discussion can work over any base field.) We let

$$X = \text{Riemann surface.}$$

On the left side of the Langlands conjecture we now have

$$\begin{aligned} \{\pi_1(X) \to GL_n(\mathbb{C})\} &= \{\text{locally constant sheaves on } X \text{ of rank } n \text{ over } \mathbb{C}\} \\ &= \{\text{holomorphic vector bundles of rank-}n \text{ on } X \text{ with a (flat) connection}\}. \end{aligned}$$

(Everything that we will be considering from now on is holomorphic.)

On the the right side of the Langlands conjecture we have

$$GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O}) = \mathrm{Bun}_n(\mathbb{C})$$

where Bun_n is the moduli space of rank-n vector bundles on X.

We think of vector bundles in terms of **lattices**. This is an abstract notion that might be unfamiliar to most physicists, so let's describe in in more detail. First, we formally define the space of lattices

$$\{\mathcal{O}_x\text{-lattices in } F_x^n\} = GL_n(F_x)/GL_n(\mathcal{O}_x).$$

This is an abstract generalization of the statement that

$$\{\text{lattices in } \mathbb{R}^n\} = GL_n(\mathbb{R})/GL_n(\mathbb{Z}).$$

Here $\mathcal{O}_x \subset F_x$ can be thought of as the "ring of integers" in the field F_x.

In order to understand better why $GL_n(F_x)/GL_n(\mathcal{O}_x)$ can be identified with the space of vector bundles, consider a vector bundle

$$\begin{tikzcd} \mathbb{C}^n \arrow[r] & E \arrow[d] \\ & X \end{tikzcd}$$

Choose a basis of rational sections (that are allowed to have poles),

$$s_1, \ldots, s_n.$$

These sections allow us to embed $E \hookrightarrow F^n$, where we recall that

$$F = \text{field of rational functions on } X.$$

(To understand the embedding $E \hookrightarrow F^n$, note that every section of E can be written as a linear combination of $\sum_{j=1}^n a_j s_j$, and the coefficient $a_j \in F$ defines the j^{th} coordinate in F^n.) Now choose a trivialization of E on all formal discs at all points $x \in X$. Each formal disc has transition functions to the rational sections. In fact, to describe the vector bundle E we need to specify, at each x, which sections are regular. (We do this by requiring certain linear combinations of residues of the n functions $a_1, \ldots, a_n$ at x to vanish.) Let

$$E_x = \{\text{sections of } E \text{ on a tiny formal disc around } x \text{ which are regular at } x\}.$$

So, at each point we get an element in $GL_n(F_x)$ that describes the transition functions from the local trivialization of E_x to the global sections. Note also that except for a finite number of x's, these transition functions are holomorphic, so in $GL_n(\mathcal{O}_x) \subset GL_n(F_x)$. Now, the collection of all the transitions functions in $GL_n(F_x)$ gives us an element in $GL_n(\mathbb{A})$. But different elements in $GL_n(\mathbb{A})$ can give equivalent sections. Multiplying by an element of $GL_n(F)$ (from the left) only changes the basis $s_1, \dots, s_n$. Multiplying by an element of $GL_n(\mathcal{O})$ (from the right) only changes the local trivialization of E_x. We conclude that

$$\mathrm{Bun}_n(\mathbb{C}) = GL_n(\mathbb{F})\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O}).$$

3.4 $\mathcal{D}$-modules

Let Y be a smooth algebraic variety over $\mathbb{C}$. Let

$$\mathcal{D}_Y = \text{sheaf of linear differential operators with algebraic coefficients on } Y.$$

We can write such an operator as

$$P = \sum a_\alpha \frac{\partial^\alpha}{\partial y^\alpha}.$$

Here α is a multi-index, and a_α's are single-valued algebraic functions (polynomials). Now let f be a function on Y, and define the **$\mathcal{D}$-module**

$$\mathcal{M} \equiv \mathcal{D} \cdot f {:=} \mathcal{D}/I,$$

where I is the ideal

$$I {:=} \{P \in \mathcal{D} | \quad Pf = 0\}.$$

If f is "good" then I is large and $\mathcal{M}$ is small. We call $\mathcal{M}$ when I is maximally overdetermined, i.e. when the space of solutions P to $Pf = 0$ is finite dimensional over $\mathbb{C}$. [???]

Example.

- $f = 1 \Longrightarrow \mathcal{D} \cdot 1 = \mathcal{O}$.
- Take $Y = \mathbb{C}$ and $f = \delta_0$ (the delta function at the origin $0 \in \mathbb{C}$)

  $$\Longrightarrow \mathcal{D} \cdot \delta_0 = \mathbb{C}\big[z, \frac{\partial}{\partial z}\big]/(z) = \mathbb{C}\big[\frac{\partial}{\partial z}\big].$$

  [But $z\frac{\partial}{\partial z}\delta(z) = -\delta(z) \neq 0$. ???]
- Hypergeometric function.

- Now suppose we are given a vector bundle with a The vector fields act on sections of the bundle, and due to the flatness condition the action extends to all of $\mathcal{D}$.

For a generic C^∞ function, I would be $\{0\}$. (Note that f is not necessarily algebraic.)

Remark 3.2. Not every $\mathcal{D}$-module can be written as $\mathcal{D}/I$, but irreducible ones can!

3.5 Pull-back and push-forward

We can **pull-back** and **push-forward** $\mathcal{D}$-modules.

pull-back	⟷	pull-back of functions,
push-forward	⟷	integrate along fiber.

Example. For the projection map $X \times Y \xrightarrow{\pi} X$, the push forward is like taking deRham cohomology on the fiber,

$$\pi_+\mathcal{M} = \pi_*\big(\mathcal{M} \underset{a_Y}{\otimes} (\Omega^0_Y \to \Omega^1_Y \to \cdots)\big)$$

Example. For the embedding $i : Y \hookrightarrow X$, with

$$X = \mathbb{C}^n, \qquad Y = \{z_1 = z_2 = \cdots = z_k = 0\},$$

the push forward is

$$i_+\mathcal{M} = \mathbb{C}[\partial_1, \dots, \partial_k] \underset{\mathbb{C}}{\otimes} \mathcal{M}.$$

Example. Consider the Gauss hypergeometric function ${}_1F_2$ that satisfies the differential 2^{nd} order differential equation

$$\big[(1-z)z\frac{d^2}{dz^2} + (\gamma - (\alpha+\beta+1)z)\frac{d}{dz} - \alpha\beta\big]{}_1F_2(z) = 0,$$

and this differential operator defines a $\mathcal{D}$-module as above. On the other hand there is an integral representation for ${}_1F_2$,

$${}_1F_2(z) = \int_0^1 \Phi(x,z)dx, \qquad \Phi(x,z) := \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{-\alpha}.$$

The integrand Φ satisfies a set of 1^{st} order differential equations

$$\Big(\frac{\partial}{\partial x} + \frac{1-\beta}{x} + \frac{\gamma-\beta-1}{1-x} - \frac{\alpha z}{1-zx}\Big)\Phi(x,z) = 0,$$
$$\Big(\frac{\partial}{\partial z} - \frac{\alpha x}{1-zx}\Big)\Phi(x,z) = 0.$$

This example is related to the **Gauss-Manin connection**. We can construct a $\mathcal{D}$-module from the solutions of some differential equations such as those satisfied by $\Phi(x,z)$. The push-forward operation constructs another $\mathcal{D}$-module by integrating over the fiber upstairs (i.e. integrating over x). We get solutions of another set of differential equations (the 2^{nd} order ones, in this example) which is the push-forward. The push-forward operation thus allows us to construct complicated modules from simpler ones.

[What is the general definition of push-forward???]

Remark 3.3. If $\mathcal{M}$ is holonomic then

$$DR\mathcal{M} = \{\mathcal{M} \otimes \Omega^0 \to \mathcal{M} \otimes \Omega^1 \to \cdots\}$$

[???] $DR\mathcal{M}$ is a **perverse sheaf**.

3.6 Cuspidality

$$GL_n = \left\{ \begin{array}{c|c} \cdots & \cdots \\ \hline \cdots & \cdots \end{array} \right\} \supset \left\{ \left(\begin{array}{c|c} L & \star \\ \hline 0 & L \end{array} \right) \right\} \equiv P.$$

P is called a **parabolic subgroup**. We have a map from P to the diagonal blocks that we denote by L. We therefore have maps

$$\begin{tikzcd} \mathrm{Bun}_n & \mathrm{Bun}_P \arrow[l, "p"'] \arrow[r, "q"] & \mathrm{Bun}_L \end{tikzcd}.$$

A module $\mathcal{M}$ on Bun_n is **cuspidal** if

$$q_* p^* \mathcal{M} = 0, \qquad \text{for all parabolic subgroups } P \subset GL_n$$

Remark 3.4. The relation of this definition to cusp forms on $SL_2(\mathbb{Z})\backslash\mathbb{U}$ is not so obvious in the language of automorphic forms, but it is more obvious if we phrase things in terms of the double coset of $GL(\mathbb{A})$. [???]

3.7 Hecke operators

We define

$$\mathrm{Hecke}_i = \{(\mathcal{L}, \mathcal{L}', x) | \quad \mathcal{L} \subset \mathcal{L}' \subset \mathcal{L}_{(x)}, \quad i = \dim(\mathcal{L}'/\mathcal{L})\}$$

Here $\mathcal{L}, \mathcal{L}' \in \mathrm{Bun}_n$ are n-dimensional vector bundles over X. $\mathcal{L}_{(x)}$ is the **twist of $\mathcal{L}$ by** (x). Note that

$$\mathcal{L}_{(x)}/\mathcal{L} = n\text{-dimensional vector space supported at } x.$$

[Here $\mathcal{L}$ probably denotes the module of sections on the vector space, not the total space!] On the level of sheaves, we have the maps

$$\begin{tikzcd}
 & \mathrm{Hecke}_i \ni (\mathcal{L},\mathcal{L}',x) \arrow[dl, "r"'] \arrow[dr, "s"] & \\
\mathcal{L}' \in \mathrm{Bun}_n & & X \times \mathrm{Bun}_n \ni (x,\mathcal{L})
\end{tikzcd}$$

We can identify the fiber,

$$s^{-1}(x,\mathcal{L}) = \{\text{space of } i\text{-planes in } \mathbb{C}^n\}.$$

The **Hecke operator** is computed as

$$T^i\mathcal{M} = s_* r^* \mathcal{M},$$

where s_* is the push-forward and r^* is the pull-back. $\mathcal{M}$ here is any $\mathcal{D}$-module on Bun_n . The push-forward operation here is an integral over the fiber which, as we have seen above, is a Grassmanian in this case.

Example. It may be instructive to look at part of the diagram involving Hecke_i and restrict it to a point. Take, for example, $n = 2$ and $i = 1$. At each point [$x \in X$] we can can modify $\mathcal{L}$ to allow the sections to have a pole at that point. [In fact, we can still require that one coordinate of the $\mathbb{C}^2$-fiber of $\mathcal{L}$ at x be regular.] There is thus a $\mathbb{P}^1$-worth of modifying the rank-2 vector bundle $\mathcal{L}$ to $\mathcal{L}_{(x)}$. [Now the push forward operation, when restricted to x, gives]

$$H^*(\mathbb{P}^1, \Omega^0_{\mathbb{P}^1} \xrightarrow{d} \Omega^1_{\mathbb{P}^1} \xrightarrow{d} \Omega^2_{\mathbb{P}^1}) = (\mathbb{C}, 0, \mathbb{C}).$$

[...] Now, if we apply the transformation $s_* r^*$ to a bundle in Bun_2 we get a new bundle in Bun_2 .

3.8 Geometric Langlands conjecture

We can now state the **geometric Langlands conjecture**

Theorem 3.2. *Given an irreducible rank n vector bundle E on X with connection, there exists an irreducible cuspidal holonomic $\mathcal{D}$-module $\mathcal{M}_E$ on* Bun_n *such that*

$$T^i\mathcal{M}_E = \wedge^i E \boxtimes \mathcal{M}_E.$$

Here $\wedge^i E$ is the i^{th} exterior power of E which lives on X, $\mathcal{M}_E$ lives on Bun_n, *and $\boxtimes$ is a* **sheaf-product**.

Remark 3.5. The connection is holomorphic and so automatically flat.

Lectures given by **David Ben-Zvi**.

3.9 Geometric Langlands conjecture for $n = 1$

We will now demonstrate how the geometric Langlands conjecture works in the special case of $n = 1$. We take X to be a Riemann surface. Take

$$L \in \{\text{Line bundles on } X \text{ equipped with a flat holomorphic connection}\}.$$

The Langlands conjecture states that there is a corresponding

$$K_L \in \{\text{automorphic } \mathcal{D}\text{-modules on } \mathrm{Bun}_1\},$$

and K_L must be an eigensheaf of Hecke operators.

We have the identification

$$\mathrm{Bun}_1 = \mathrm{Pic}\, X \equiv \text{Picard group of } X = \text{Space of holomorphic line bundles.}$$

We have the mapping

$$\mathrm{Jac}\, X \xrightarrow{\text{Jacobian}} \mathrm{Pic}\, X \xrightarrow{\deg} \mathbb{Z},$$

where

$$\mathrm{Jac}\, X = H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) = H^{0,1}(X)/H^1(X, \mathbb{Z})$$

is the **Jacobian** of the Riemann surface, which is a torus of complex dimension $g = \mathrm{genus}\, X$. (Note that $H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$.) We therefore have the identification

$$\mathrm{Pic}\, X \simeq \mathrm{Jac}\, X \times \mathbb{Z}.$$

Note that

$$\pi_1(\mathrm{Jac}\, X) = H^1(X, \mathbb{Z}).$$

3.10 Constructions of the Langlands correspondence

Take a representation

$$L : \pi_1(X) \to \mathbb{C}^* \equiv GL_1(\mathbb{C}).$$

This is the monodromy of the line bundle.

$$\begin{tikzcd}
L:\,\pi_1(X) \arrow[r] & \mathbb{C}^* & \pi_1(\operatorname{Jac} X) \arrow[l] \\
H_1(X,\mathbb{Z}) \arrow[ur] \arrow[r, phantom, "\simeq"] & H^1(X,\mathbb{Z}) \arrow[u, dashed] \arrow[ur, equal] &
\end{tikzcd}$$

where we used Poincaré duality to set $H_1(X,\mathbb{Z}) \simeq H^1(X,\mathbb{Z})$. Thus, from L we got a map

$$K_L : \pi_1(\operatorname{Jac} X) \to \mathbb{C}^*.$$

Here K_L is a line bundle with flat connection on $\operatorname{Jac} X$. We can identify it with a $\mathcal{D}$-module.

Remark 3.6. If we replace GL_1 with a torus $T = \mathbb{C}^* \underset{\mathbb{Z}}{\otimes} \Lambda$ for some lattice Λ (an actual lattice this time, as opposed to the abstract lattice that corresponded to a vector bundle before), we get

$$\operatorname{Bun}_T = \operatorname{Pic} X \underset{\mathbb{Z}}{\otimes} \Lambda \cong (\operatorname{Jac} X \times \mathbb{Z})^{\operatorname{rank}\Lambda} = \{\text{holomorphic } T\text{-bundles on } X\}$$

$$\begin{tikzcd}
\pi_1(X) \arrow[rr] \arrow[dr] & & T^\vee = \mathbb{C}^* \otimes \Lambda^\vee & \Longleftrightarrow & \pi_1(\operatorname{Bun}_T^0) \arrow[rr] \arrow[dr, equal] & & C^* \\
& H_1(X,\mathbb{Z}) \arrow[ur] & & & & H_1(X,\mathbb{Z}) \otimes \Lambda \arrow[ur] &
\end{tikzcd}$$

Here Bun^0 is the connected component of Bun.

How can we extend K_L from $\operatorname{Jac} X$ to $\operatorname{Pic} X$? For a generic $x \in X$ we define the operator

$$H_x : \operatorname{Pic}^n X \xrightarrow{\sim} \operatorname{Pic}^{n+1} X$$
$$\mathcal{L} \mapsto \mathcal{L}_{(x)}$$

Here Pic^n is the line bundles of degree n, and $\mathcal{L}_{(x)}$ s a modification of the line bundle $\mathcal{L}$ at x whereby we allow the sections to havea 1^{st} order pole at x. [For nongeneric x, if the sections of $\mathcal{L}$ are allowed to have a k^{th} order pole, then the modification $\mathcal{L}_{(x)}$ allows them to have a $(k+1)^{th}$ order pole.]

3.11 Hecke eigensheaf condition

We would like to relate

$$K_L|_{H_x\mathcal{L}} \xrightarrow{\sim} L|_x \otimes K_L|_{\mathcal{L}}.$$

Here $L|_x$ is the fiber of $\mathcal{L}$ at x, which is a complex line. [Recall that L is the fixed line bundle from which we constructed the "line bundle over the space of line bundles" K_L, and $\mathcal{L}$ is an arbitrary coordinate on the "space of line bundles" Bun_1.] K_L should be a $\mathcal{D}$-module on $\mathrm{Pic}\, X$, but we need a compatibility condition between the components [that is, we can arrive at the same line-bundle over X by applying the operators H_x at various points in different order, i.e. $H_x \circ H_{x'}\mathcal{L} = H_{x'} \circ H_x\mathcal{L}$. The fiber of K_L at $H_x H_{x'}\mathcal{L}$ should be independent of the order of H_x and $H_{x'}$.]

$$H_x^* \cdot K_L \xrightarrow{\sim} L_x \cdot K_L \qquad \text{on } \mathrm{Pic}^n . \text{ [???]}$$

$$\begin{tikzcd}
& \mathrm{Hecke}_x = \{(\mathcal{L}, \mathcal{L}', \mathcal{L}|_{X-x} \xrightarrow{\sim} \mathcal{L}'|_{X-x}\} \arrow[dl, "r"'] \arrow[dr, "s"] & \\
\mathcal{L} \in \mathrm{Pic}\, X & & \mathcal{L}' \in \mathrm{Pic}\, X
\end{tikzcd}$$

Here $\mathcal{L}|_{X-x} \xrightarrow{\sim} \mathcal{L}'|_{X-x}$ is an isomorphism of the line bundles $\mathcal{L}$ and $\mathcal{L}'$ away from the point x. Hecke_x has has a simple description.

$$\begin{tikzcd}
& \mathrm{Hecke}_x = \mathrm{Pic}\, X \times \mathbb{Z} = \{\mathcal{L}, n\} \arrow[dl, "r"'] \arrow[dr, "s"] & \\
\begin{array}{l} \mathrm{Pic}\, X \\ \ \mathcal{L} \end{array} & & \begin{array}{l} \mathrm{Pic}\, X \\ \mathcal{L}(nx) \end{array}
\end{tikzcd}$$

where we assign to a line bundle $\mathcal{L} \in \mathrm{Pic}\, X$ the line bundle $\mathcal{L}(nx)$ which is modified by [a divisor] nx.

$$\underbrace{\mathcal{L} \subset \mathcal{L}'}_{1-dim} \subset \mathcal{L}(x).$$

Now vary x, to get

$$\begin{tikzcd}
& \mathrm{Hecke}_x = \mathrm{Pic}\, X \times \mathbb{Z} \times X = \{\mathcal{L}, n, x\} \arrow[dl, "r"'] \arrow[dr, "s", "\mathbb{Z}"'] & \\
\mathrm{Pic}\, X & & \mathrm{Pic}\, X \times X
\end{tikzcd}$$

The Hecke eigensheaf condition is $s_* r^* K_L \xrightarrow{\sim} K_L \boxtimes L$. The fiber of $K_L \boxtimes L$ at $x \in X$ is $K_L \otimes L|_x$. K_L has a flat connection on $\operatorname{Pic} X$ and L has a float connection on X. We were looking at

$$\operatorname{Pic} X \longleftarrow \operatorname{Pic} X \times X \times \{1\} \rightarrow \operatorname{Pic} X \times X.$$

(Here $\{1\} \in \mathbb{Z}$.)

3.12 Composition of Hecke operators

Start with

$$K_L|_{\mathcal{L}(x_1+x_2+\cdots+x_n)} = H_{x_1} \cdot H_{x_2} \cdots H_{x_n} \cdot K_L|_{\mathcal{L}}$$

This is reminiscent of an eigenvalue equation

$$H_x(K_L) \rightarrow L_x \otimes K_L.$$

$$K_L|_{\mathcal{L}(x_1+x_2+\cdots+x_n)} \xrightarrow{\sim} L|_{x_1} \otimes L|_{x_2} \otimes \cdots \otimes L|_{x_n} \otimes K_L.$$

are the eigenvalues. If we know K_L on a particular line-bundle, say $K_L|_{\mathcal{O}} = \mathbb{C}$, [where $\mathcal{O}$ is the trivial line-bundle ???] then we can calculate it at every effective divisor,

$$K_L|_{\mathcal{O}(x_1+x_2+\cdots x_n)} = L|_{x_1} \otimes L|_{x_2} \otimes \cdots \otimes L|_{x_n}.$$

So, starting with the trivial bundle $\mathcal{O}$, we can modify it to $\mathcal{O}(x)$ for all $x \in X$. In this we we get a copy of the curve in $\operatorname{Pic} X$.

$$X \longrightarrow \operatorname{Pic} X$$

$$x \mapsto \mathcal{O}(x).$$

[Let $\Sigma \subset \operatorname{Pic} X$ be the image of X, then]

$$K_L|_{\Sigma} = L.$$

Similarly,

$$\operatorname{Sym}^n X \longrightarrow \operatorname{Pic}^n X$$

$$x_1 + \cdots + x_n \mapsto \mathcal{O}(x_1 + \cdots x_n).$$

where Sym is the symmetric product. Now instead of L on X we have $\operatorname{Sym}^n L$ which is a flat line-bundle on $\operatorname{Sym}^n X$. Does it come from something on $\operatorname{Pic}^n X$? This is the Hecke condition. The key observation is (Deligne) that $\mathbb{P}^n$ is simply connected and therefore for $n \geq g$ we have a surjective map $\operatorname{Sym}^n X \twoheadrightarrow \operatorname{Pic}^n X$, and for $n > 2g-2$ $\operatorname{Sym}^n X \rightarrow \operatorname{Pic}^n X$ is a fiber-bundle with fibers $\mathbb{P}^{n-g}$. Take $\operatorname{Sym}^N L = \pi^* K_L$. The fibers are simply connected so $\operatorname{Sym}^n L$ cannot do anything interesting on the fiber. We have thus satisfied our Hecke condition.

More explanation by Ed Frenkel

[We can define,]

$$K_L|\mathcal{O}(x_1 + \cdots + x_n - y_1 - \cdots - y_m) \simeq L|_{x_1} \otimes \cdots L^*|_{y_1} \otimes \cdots L^*|_{y_m}.$$

But this is overdetermined. If $D = (f) = \sum x_i - \sum y_i$ is the divisor of a meromorphic function then

$$L|_{x_1} \otimes \cdots \otimes L|_{x_n} \simeq L|_{y_1} \otimes \cdots \otimes L|_{y_m}$$

are canonically identified. But this requirement follows from $\pi_1(\mathbb{P}^N) = 0$. For finite fields this gives a reciprocity law.

Comment by Ron Donagi

Here is an alternative to Deligne's proof that works for $\mathbb{C}$: every local system on the Jacobian is translationally invariant!

Lectures given by **Ed Frenkel**.

4 Tuesday morning session

4.1 Summary of yesterday

X is a smooth compact curve over $\mathbb{C}$. We discussed the geometric Langlands conjecture for GL_n

$$\left\{\begin{array}{ll} \mathcal{E}=(\mathcal{F},\nabla) & -\text{rank } n \text{ bundle on } X \\ & \qquad\qquad \text{with holomorphic flat connection.}\end{array}\right\} \Longleftrightarrow \left\{ \mathcal{M}_{\mathcal{E}} \quad -\mathcal{D}\text{-module on } \mathrm{Bun}_n \right\}$$

where Bun_n is the moduli space of holomorphic rank-n bundles on X, required to have a Hecke eigensheaf with "eigenvalue" $\mathcal{E}$.

In the classical theory we consider $GL_n(F)GL_n(\mathbb{A})/GL_n(\mathcal{O})$. We have the **Hecke eigensheaf** condition:

$$\mathrm{Hecke}_i(\mathcal{M}_{\mathcal{E}}) \simeq \wedge^i E \boxtimes \mathcal{M}_{\mathcal{E}}, \qquad i=1,\dots,n.$$

Recall:

$$\begin{tikzcd} & \{(\mathcal{L}',x,\mathcal{L})| \quad \mathcal{L}\subset\mathcal{L}'\subset\mathcal{L}(x), \quad \dim\mathcal{L}'/\mathcal{L}=i\} = \mathrm{Hecke}_i \subset \mathrm{Bun}_n \times(X\times\mathrm{Bun}_n)\} \arrow[dl, "r"'] \arrow[dr, "s"] & \\ \mathcal{L}'\in\mathrm{Bun}_n & & (x,\mathcal{L})\in X\times\mathrm{Bun}_n \end{tikzcd}$$

$\mathcal{L}'$ is obtained from modification of $\mathcal{L}$ at one point. This modification could be a pole at x along a particular line, for example.

Fibers of both objects are **Grassmanians**. Given the diagram, we get an analog of an integral transform [???] whose kerlnel is the set of characteristic functions of the incidence variety.

$$\mathrm{Hecke}_i(\mathfrak{g}){:=}s_*r^*(\mathfrak{g}), \qquad \text{for any } \mathcal{D}\text{-module } \mathfrak{g} \text{ on } \mathrm{Bun}_n.$$

Here $s_*r^*(\mathfrak{g})$ is a $\mathcal{D}$-module on $X\times\mathrm{Bun}_n$. It is $\wedge^i\mathcal{E}\times\mathcal{M}_{\mathcal{E}}$.

Example

Take the trivial bundle. Then,

$$\mathcal{M}_{\mathcal{E}}=\mathcal{O}_{\mathrm{Bun}_n}, \qquad \mathrm{Hecke}_i(\mathcal{O})=\wedge^i\mathcal{E}\boxtimes\mathcal{O}, \qquad \mathcal{E}_0=\bigoplus_{j=1}^{n}\mathcal{E}_{0,j},$$

where $\mathcal{E}_{0,j}$, is a trivial rank-1 local system on X in cohomological dimension $2(j-1)$.

References

[Deligne-Grothendik] Deligne and Grothendik.

[Atkins-Shimura] Atkins and Shimura.

[Deligne] Lectures by P. Deligne, Arizona Winter School 2002,
`http://www-math.mit.edu/ kedlaya/papers/deligne-notes.pdf`

[Sheaves] Lecture notes on sheaves,
`http://www.math.jussieu.fr/ schapira/polycopies/Shv.pdf`