Langlands - Physics

EDWARD DAY 2 - Afternoon.

$\Gamma(\mathrm{Bun}_G, D_{-h^\vee}) \simeq \mathbb{C}[D_i]_{i=1\ldots B_v}$

$\chi : \mathbb{C}[D_i] \to \mathbb{C}$

Claim: Each S_χ is a Hecke eigensh on Bun_G with the eigenvalue being a flat bdle on X. det by χ.

Geom Langlands for general G.

$\left\{ \begin{array}{l} \mathcal{E} = (F, \nabla) - {}^LG\text{-bndle} \\ \text{on } X \text{ with a hol. flat} \\ \text{conn} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} D\text{-mod} \\ \mathrm{Bun}_G \end{array} \right.$

We are assuming G s.c.

G s-conn. $\leftrightarrow$ ${}^LG$ - adjoint

$SL_n \leftrightarrow PGL_n$.

Need to define Hecke operators

The Hecke operators are functors.

$\left[\begin{array}{l} D\text{-mod} \\ \text{on } \mathrm{Bun}_G \end{array} \right] \longrightarrow \left[\begin{array}{l} \text{Complexes of} \\ D\text{-modules on} \\ X \times \mathrm{Bun}_G \end{array} \right.$

one labeled by the dominant integral wts of ${}^LG$ (e.g. dom int. coweights of G)

$\lambda \in {}^LP^+ \rightsquigarrow H_\lambda$

For $GL(n)$ $H_i = H_{\omega_i}$ $i = 1, \dots, n-1$
$H_n = H_{\det}$

Q: for GL_n suffices to check H_1 ??

Want $H_\lambda(M_\mathcal{E}) \xrightarrow{\sim} V_{\lambda,\mathcal{E}} \boxtimes M_\mathcal{E}$

Eigenprop $\longrightarrow$ $\mathcal{E} \overset{{}^L G}{\times} V_\lambda$

(B-D) Claim: 1) Spec $\Gamma(Bun_G, D_{h^\vee})$ is a subspace of the space of ${}^L G$-flat bdles on X called the space of ${}^L G$-opers on X.
2) S_χ is a Hecke eigensheaf with eigenvalue χ.

Let $\mathcal{N}$ be a coherent sheaf on spec $D_{-h^\vee}$ where $\mathbb{D}_{-h^\vee} = \Gamma(Bun_G, \mathcal{D}_{-h^\vee})$ define a D-mod on Bun_G

$$S_\mathcal{N} = \mathcal{D}_{-h^\vee} \underset{\mathbb{D}_{-h^\vee}}{\otimes} \mathcal{N}$$

So get functor $\begin{bmatrix} \mathcal{O}\text{-mod on} \\ \text{Spec } D_{-h^\vee} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{D}_{-h^\vee}\text{-mo} \end{bmatrix}$

What is Spec $\mathcal{D}_{-h^\vee}$?

For $G = SL_2$, ${}^LG = PGL_2$
this is the space of flat ${}^LG$ - bdles
of the form $(\mathcal{F}_0, \nabla)$

$$0 \to \Omega^{1/2} \to \mathcal{F}_0 \to \Omega^{-1/2} \to 0$$

$\rightsquigarrow$ this defines $\mathcal{F}_0$ unambiguous
as PGL_2 bdl.

In general, also all flat bdles
of the form $(\mathcal{F}_{0, {}^LG}, \nabla)$ where
$\mathcal{F}_{0, {}^LG}$ is an ${}^LG$ - bundle induced
by $\mathcal{F}_0$ under the principal emb $PGL_2 \to {}^LG$

$Op(G)$ has $H^4(X, Z(G))$ - compon

these are very ined local systems..
maximally ined. in a sense.

<u>Concrete example</u>: A GL_n - oper
on X is a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$
where $\mathcal{F}$ - rk n hol. v.b., ∇ - hol.
connection ($(\mathcal{F}, \nabla)$ - hol flat)
and $\mathcal{F}_B$ is a flag
$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_n = \mathcal{F}$$

$$\Rightarrow \quad \nabla(F_i) \subset F_{i+1} \otimes \Omega^1_X$$

$$F_i/F_{i-1} \xrightarrow[\sim]{\nabla} \left(F_{i+1}/F_i\right) \otimes \Omega^1_X .$$

for $i = 1, \dots, n-1$.

<u>Trivialize F locally</u> compatibly with F_B choose a local coord z on X

$$\nabla = \partial_z + \begin{pmatrix} * & * & & & * \\ -1 & * & & & \\ & -1 & & & \\ & & & & * \\ & 0 & \ddots & & \\ & & & -1 & * \end{pmatrix}$$

put -1

changing triv. compatibly with flag = conj. by upper-Δ matrix w. 1's on diagonal.

In each gauge equivalence class can pick a unique rep.

$$\partial_z + \begin{pmatrix} v_0 & v_1 & \cdots & v_{n-2} & v_{n-1} \\ -1 & & & 0 & \\ & \ddots & & & \\ & & \ddots & & \\ & & & -1 & \end{pmatrix}$$

$\partial_z \rightsquigarrow$ same as diff op.

$$\partial_z^n + v_0 \partial_z^{n-1} + \cdots \; v_{n-1}$$

this is op $\Omega^{-\left(\frac{n-1}{2}\right)} \longrightarrow \Omega^{\left(\frac{n+1}{2}\right)}$

for PGL_n oper, we choose a v.b. up to $\otimes$ with a line bundle. We can take the determinant to be trivial.

$$\partial_z + \begin{pmatrix} 0 & v_1 & \cdots & v_{n-1} \\ -1 & & & \\ & \ddots & & \\ & & -1 & \end{pmatrix}.$$

For $n=2$, PGL_2 - oper is $\partial_z^2 + v(z) : \Omega^{-1/2} \to \Omega^{3/2}$ - proj. connection transform like the stress tensor $T(z)$

Observe that $\mathcal{O}p_{PGL_2} = \mathcal{O}p_{PGL(2)}$ is a torsor over $H^0(X, \Omega^{\otimes 2})$
So an affine space.

$$\mathcal{O}p_{^LG}(X) \cong \mathrm{Proj}(X) \times \bigoplus_{i=2}^{\ell} H^0(X, \Omega^{\otimes(d_i+1)})$$

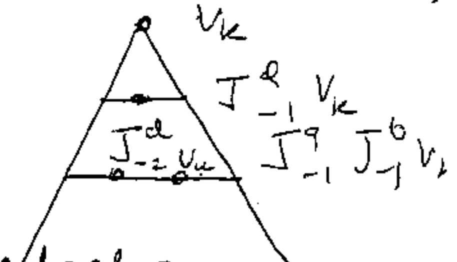
(d_i+1) - order of ith Cas. ops. $Z(U(\mathfrak{g})) \simeq \mathbb{C}[C_i]_{i=1,\dots,\ell}$.

$$\dim \mathcal{O}p_{^LG} = 3g-3 + \sum_{i=2}^{\ell} (2d_i+1)(g-1)$$
$$= \underbrace{\sum (2d_i+1)}_{} (g-1)$$
$$= \dim G. \quad = \dim \mathrm{Bun}_G$$

$$\left[\partial_z + \begin{pmatrix} q_1 & q_2 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad f_1' = q_1 f_1 + q_2 f_1 \qquad f_2' = f_1$$

Revisit $V_k(\mathfrak{g})$: $\{J^a\}$ – basis of $\mathfrak{g}$ (assume orthonormal w.r.t $\langle\,\rangle$)

$$J^a_n = J^a \otimes t^n \in \hat{\mathfrak{g}}.$$



Have chiral (or vertex) algebra structure on $V_k(\mathfrak{g})$

$A \in V_k(\mathfrak{g}) \longrightarrow Y(A, z) = \sum A_{(n)} z^{-n-1}$

$A_{(n)}$ – operator on $V_k(\mathfrak{g})$ or any other $\hat{\mathfrak{g}}$ – module of level k.

$$Y(v_k, z) = \mathrm{Id}$$

$$Y(J^a_{-1} v_k, z) = \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1} = J^a(z)$$

$$Y(J^a_{-2} v_k, z) = \partial_z J^a(z)$$

$$Y(J^a_{-1} J^b_{-1} v_k, z) = :J^a(z) J^b(z):$$

$$\boxed{Y(A, z)\, v_k \big|_{z=0} = A}$$

$$S = \frac{1}{2} \sum_{a = \dim \mathfrak{g}}^{\dim(\mathfrak{g})} (J^a_{-1})^2 v_k.$$

$$Y(S, z) = \sum_{a=1} :J^a(z)^2: = \sum_{n \in \mathbb{Z}} S_n z^{-n-2}$$

$\underline{k = -h^\vee}$ $\quad T(z) = \dfrac{1}{k + h^\vee} S(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$

$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3-n)c_k\delta_{n,-m}$
$[L_n, J^a_m] = -m J^a_{n+m}$.

L_n's + J^a_m's give action of
$\mathrm{Vir} \ltimes \hat{\mathfrak{g}}$ on $V_k(\mathfrak{g})$.

$J^a(z)\,dz$ - 1-form on $D^\times_z$ $(J(a)z = J^a_n$
$T(z)$ almost quadratic diff $\to$ Schwarz. deriv.
$T(z) = T(w)\left(\frac{dw}{dz}\right)^2 + \frac{c_k}{12}\{w;z\}$

$-\frac{c_k}{6}\partial_z^2 - T(z)(dz)^2 : \Omega^{-1/2} \to \Omega^{3/2}$ is coordinate-independent

$[S_n, J^a_n] = (-k+h^\vee)\, m J^a_{n+m} = 0$ if $k = -h^\vee$.

S_n's are central elements in a completed enveloping algebra of $\hat{\mathfrak{g}}$ at $k = -h^\vee$. The chiral algebra of fields contains a center
$\mathbb{C}[S(z), \partial_z S(z), \dots]$ for $\mathfrak{g} = \mathfrak{sl}_2$
this is the entire center.

[Recall that in f-d. $Z(U(\mathfrak{g}))$
$= \mathbb{C}[C_1, \dots, C_\ell]$
$\sum (J^a)^2$ ↑ Casir.